Introduction to the Finite Element Method

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Outline

First definitions / reminder ☺

Formulation of the problem

Finite element method

Numerical resolution

Application: ophthalmology

Functional spaces

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega$. We define the following functional spaces:

- $L^2(\Omega) = \left\{u : \Omega \to \mathbb{R} \mid \int_\Omega |u|^2 \mathrm{d}x < +\infty\right\}$, the space of square-integrable functions on Ω.
- $H^1(\Omega) = \left\{u \in L^2(\Omega) \,\middle|\, \frac{\partial u}{\partial x_i} \in L^2(\Omega) \text{ for all } i = 1, \ldots, d\right\}$, the Sobolev space of functions with square-integrable first derivatives.
- $H_0^1(\Omega) = \left\{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega\right\}$, the subspace of $H^1(\Omega)$ whose functions vanish on the boundary.

They have infinite dimension.

We also define the norms associated with these spaces:

- L^2 norm: $\|u\|_{L^2(\Omega)} = \sqrt{\int_\Omega |u|^2 \mathrm{d}x}$
- H^1 norm: $\|u\|_{H^1(\Omega)} = \sqrt{\|u\|^2_{L^2(\Omega)} + \sum_{i=1}^d \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(\Omega)}}$

Tiny remark on derivative

- In the definition of $H^1(\Omega)$, the derivatives $\frac{\partial u}{\partial x_i}$ are understood in the **weak sense** (distributional derivative).
- A function $u \in L^2(\Omega)$ has a weak derivative $\frac{\partial u}{\partial x_i} \in L^2(\Omega)$ if there exists a function $v \in L^2(\Omega)$ such that for all test functions $\varphi \in C_c^\infty(\Omega)$ (smooth functions with compact support):

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \mathrm{d}x = -\int_\Omega v\varphi \mathrm{d}x$$

In this case, we write $v = \frac{\partial u}{\partial x_i}$.

Hilbert spaces

Definition

A *Hilbert space* is a complete inner product space, i.e., a vector space $\mathcal{H}$ equipped with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ such that every Cauchy sequence in $\mathcal{H}$ converges to an element in $\mathcal{H}$.

Theorem

The spaces $L^2(\Omega)$, $H^1(\Omega)$, and $H^1_0(\Omega)$ are Hilbert spaces with the inner products defined as follows:

- L^2 *inner product:* $\langle u, v \rangle_{L^2(\Omega)} = \int_\Omega uv\mathrm{d}x$
- H^1 *inner product:* $\langle u, v \rangle_{H^1(\Omega)} = \langle u, v \rangle_{L^2(\Omega)} + \sum_{i=1}^{d} \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle_{L^2(\Omega)}$

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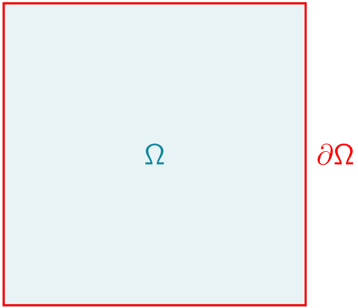
Position of the problem

- We have a bounded domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$.
- We consider the problem of **heat transfer** in Ω given by the **partial differential equation** (PDE):

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

  With $f \in L^2(\Omega)$ is a given source term, and u is the unknown temperature field.
- Δ is the **Laplace operator**, defined as the divergence of the gradient:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$



Variational formulation

- We multiply the PDE by a test function $v \in H_0^1(\Omega)$ and integrate over Ω:

$$\int_\Omega (\Delta u) v \mathrm{d}x = -\int_\Omega f v \mathrm{d}x$$

- Using integration by parts (Green's identity), we have:

$$-\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \int_{\partial\Omega} (\nabla u \cdot n) v \mathrm{d}\sigma = -\int_\Omega f v \mathrm{d}x$$

- Since $v = 0$ on $\partial\Omega$, the boundary term vanishes, leading to the variational formulation: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = \ell(v) \quad \forall v \in H_0^1(\Omega)$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \quad \ell(v) = \int_\Omega f v \mathrm{d}x$$

Existence of a solution

Theorem: Lax–Milgram

Let V be a Hilbert space. Let $a\colon V \times V \to \mathbb{R}$ be a bilinear form that is :

- **Continuous**: There exists a constant $C > 0$ such that $|a(u, v)| \leqslant C\|u\|_V\|v\|_V$ for all $u, v \in V$.
- **Coercive**: There exists a constant $\alpha > 0$ such that $a(u, u) \geqslant \alpha\|u\|_V^2$ for all $u \in V$.

Let $\ell\colon V \to \mathbb{R}$ be a continuous linear functional ($\exists C > 0$ such that $|\ell(v)| \leqslant C\|v\|_V$ for all $v \in V$).

Then there exists a unique $u \in V$ such that

$$a(u, v) = \ell(v) \quad \forall v \in V$$

Checking the hypotheses of Lax–Milgram

For our problem:

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \quad \ell(v) = \int_\Omega f v \mathrm{d}x$$

- **Continuity of a and ℓ**: follows from Cauchy–Schwarz — an integral of a product is controlled by the product of L^2 norms. No surprise here.

Checking the hypotheses of Lax–Milgram

For our problem:

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad \ell(v) = \int_\Omega f v dx$$

- **Coercivity of** a is the one real question: does $a(u, u) = \|\nabla u\|_{L^2}^2$ control the *full* norm $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$?
- At first glance: no — a only sees the gradient, not u itself.
- **This is exactly where** $u = 0$ **on** $\partial\Omega$ **saves us**: the **Poincaré inequality** says that on $H_0^1(\Omega)$, a function is controlled by its own gradient:

$$\boxed{\exists\, C_\Omega > 0, \quad \|u\|_{L^2(\Omega)} \leqslant C_\Omega \|\nabla u\|_{L^2(\Omega)} \qquad \forall u \in H_0^1(\Omega)}$$

- Intuitively: if u vanishes on the boundary, it can't be large anywhere without its gradient being large too.

So for our problem, there exists a unique solution $u \in H_0^1(\Omega)$ such that

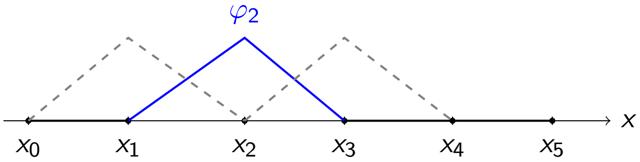
$$a(u, v) = \ell(v) \quad \forall v \in H_0^1(\Omega)$$

Finite element discretization (in 1D)

We define a finite dimensional subspace $V_N \subset V = H_0^1(\Omega)$.

- Discretize $\Omega = [0, L]$ with nodes $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = L$
- Define $V_N = \left\{ v \in C^0(\overline{\Omega}) \;:\; v|_{[x_i, x_{i+1}]} \text{ affine}, \; v(0) = v(L) = 0 \right\}$
- **Notation:** $\mathbb{P}_1(\Omega)$ is the space of piecewise affine functions on Ω
- Basis: the **hat functions** $(\varphi_i)_{i=1,\dots,N}$, one per interior node, satisfying

$$\varphi_i(x_j) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ else} \end{cases}$$



Finite element discretization (in 1D)

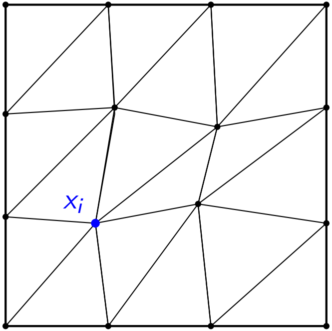
- $(\varphi_i)_{i=1,\dots,N}$ is a **basis** of V_N: it is free and spanning by construction, so $\dim V_N = N$
- **Nodal interpolation**: since $\varphi_i(x_j) = \delta_{ij}$, any $v_N \in V_N$ writes uniquely as

$$v_N = \sum_{i=1}^{N} v_N(x_i)\,\varphi_i$$

i.e. the coefficients are simply the values of v_N at the nodes

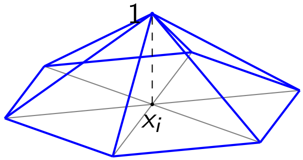
Finite element space in 2D

Discretize Ω by a **triangulation** $\mathcal{T}_h$ with nodes $(x_i)_{i=1,\dots,N}$:



- $V_N = \{v \in C^0(\overline{\Omega}) : v|_T \text{ affine on each triangle } T,\ v = 0 \text{ on } \partial\Omega\}$
- $\dim(V_N) = N$ (one basis function per interior node)

Shape of a 2D basis function φ_i



- φ_i is the **pyramid (tent) function**: value 1 at x_i, affine on each surrounding triangle, 0 on the boundary of the patch and beyond
- Same properties as in 1D: $\varphi_i(x_j) = \delta_{ij} \ldots$

Algorithm to solve the variational problem

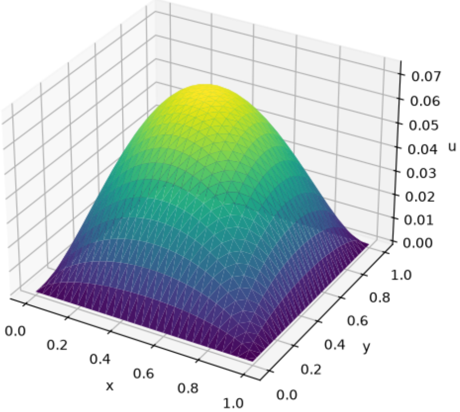
1. Discretize the domain Ω into a mesh of elements (triangles in 2D, tetrahedra in 3D).
2. Choose a finite element space V_h and construct the basis functions $\{\varphi_i\}$.
3. Assemble the stiffness matrix $\underline{\underline{\boldsymbol{A}}} \in \mathbb{R}^{N \times N}$ and load vector $\underline{\boldsymbol{b}} \in \mathbb{R}^N$:

$$\underline{\underline{\boldsymbol{A}}}_{ij} = a(\varphi_j, \varphi_i) = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \mathrm{d}x,$$
$$\underline{\boldsymbol{b}}_i = \ell(\varphi_i) = \int_\Omega f\varphi_i \mathrm{d}x.$$

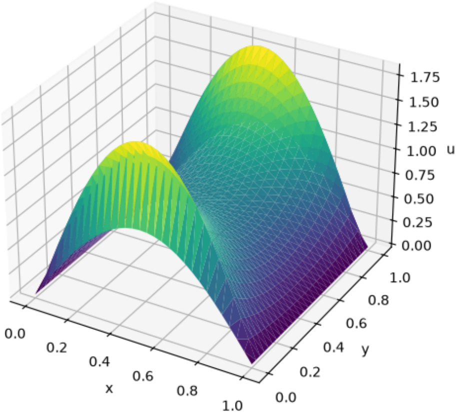
4. Solve the linear system $\underline{\underline{\boldsymbol{A}}}\mathbf{u} = \underline{\boldsymbol{b}}$ for the coefficients $\mathbf{u} = (u_1, u_2, \ldots, u_N) \in \mathbb{R}^N$.

First results

$-\Delta u = 1$ in Ω, $u = 0$ on $\partial\Omega$



$-\Delta u = 0$ in Ω, $u = \sin(\pi x)$ on $\partial\Omega$



Convergence of the method

Convergence of the finite element method

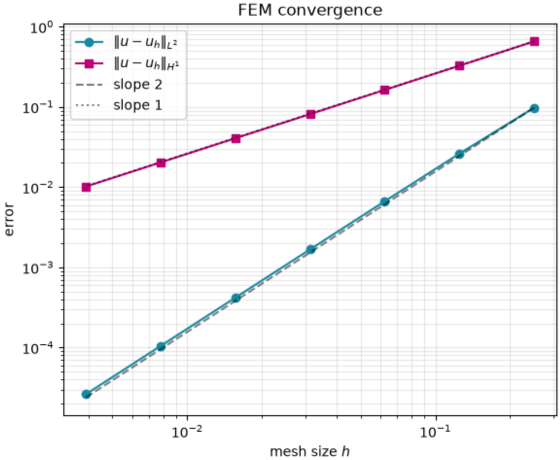
The theoretical convergence rate for piecewise linear elements is $\mathcal{O}(h)$ in the H^1 norm and $\mathcal{O}(h^2)$ in the L^2 norm, where h is the maximum element diameter.

- The function $u_{\text{exact}}(x, y) := \sin(\pi x)\sin(\pi y)$ is the exact solution to the problem:

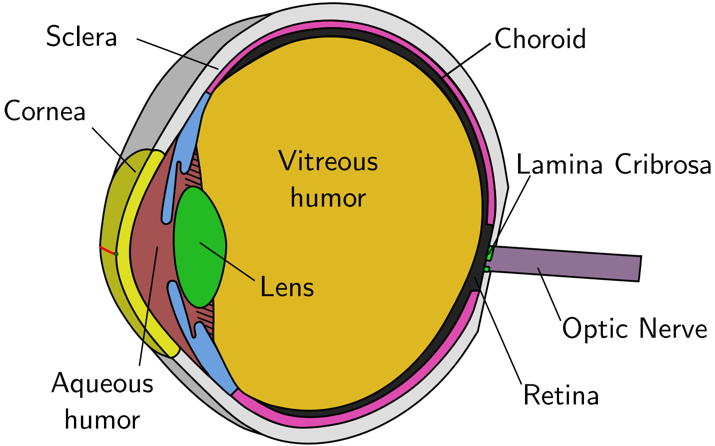
$$\begin{cases} -\Delta u = 2\pi^2 \sin(\pi x)\sin(\pi y) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

- We compute the numerical solution u_h using the finite element method on a sequence of refined meshes.
- As we picked u_{exact}, we can measure the error directly.

Convergence of the method



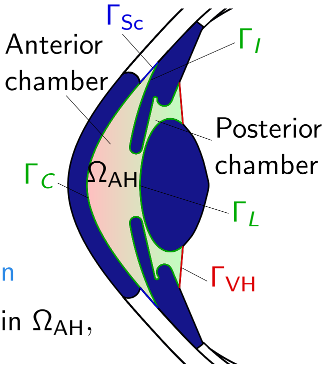
Geometry of the eye



$$\Omega = \bigcup_i \Omega_i$$

Biophysical model

- We study the **flow** of the aqueous humor in the anterior chamber of the eye, coupled with the **heat transfer** in the eye.
- The steady flow of the aqueous humor is governed by the Navier–Stokes equations:



Navier-Stokes equations Boussinesq approximation

$$\boxed{\rho(\vec{u}\cdot\nabla)\vec{u} - \nabla\cdot(2\mu\mathbf{D}(\vec{u}) - p\mathbf{I})} = \boxed{-\rho\beta(T - T_{\mathrm{ref}})\vec{g}} \quad \text{in } \Omega_{\mathrm{AH}},$$

Incompressibility
$$\nabla\cdot\vec{u} = 0 \quad \text{in } \Omega_{\mathrm{AH}},$$

Heat transfer equation
$$\rho C_p \vec{u}\cdot\nabla T - k_i \Delta T = 0 \quad \text{in } \Omega = \bigcup_i \Omega_i.$$

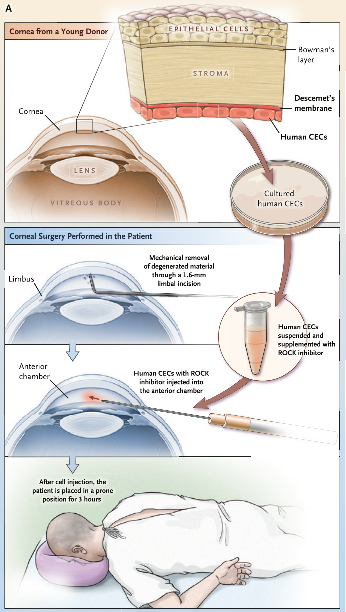
Three unknown quantities: *(i)* the **fluid velocity** $\vec{u}$, *(ii)* its **pressure** p, and *(iii)* the **temperature** of the eye T.

Motivation: Endothelial cells sedimentation

- AH flow plays a significant role in heat distribution, and intraocular pressure[a].
- Medical application: **corneal endothelial cell sedimentation** in the cornea[b].

[a] Dvoriashyna *et al.* *Ocular Fluid Dynamics*. (2019)

[b] Kinoshita *et al.* *N Engl J Med*. (2018) (Figure extracted from this reference)



Weak formulation

We define the following functional spaces :

1. The *velocity* space $\boldsymbol{V} := [H_0^1(\Omega_{\mathrm{AH}})]^3$, consisting of vector fields $\vec{u}$ with square-integrable derivatives with a trace that vanishes on the boundary;
2. the *pressure* space $Q := L_0^2(\Omega_{\mathrm{AH}}) = \left\{ p \in L^2(\Omega_{\mathrm{AH}}) \middle| \int_{\Omega_{\mathrm{AH}}} p \, \mathrm{d}x = 0 \right\}$, consisting of square-integrable scalar fields with zero mean; and
3. the *temperature* space $W := H^1(\Omega)$.

Weak formulation

Find $(\vec{u}, p, T) \in \boldsymbol{V} \times Q \times W$ such that for all $(\vec{v}, q, \varphi) \in \boldsymbol{V} \times Q \times W$:

$$\rho \int_{\Omega_{\mathrm{AH}}} (\vec{u} \cdot \nabla)\vec{u} \cdot \vec{v} \mathrm{d}x + \mu \int_{\Omega_{\mathrm{AH}}} \mathbf{D}(\vec{u}) : \nabla \vec{v} \mathrm{d}x - \int_{\Omega_{\mathrm{AH}}} p \cdot \nabla \vec{v} \mathrm{d}x$$

$$+ \int_{\Omega_{\mathrm{AH}}} \rho_0 \beta T \vec{g} \cdot \vec{v} \mathrm{d}x = \int_{\Omega_{\mathrm{AH}}} \rho_0 \beta T_{\mathrm{ref}} \vec{g} \cdot \vec{v} \mathrm{d}x,$$

$$\int_{\Omega_{\mathrm{AH}}} \nabla \cdot \vec{u} q \mathrm{d}x = 0,$$

$$\rho C_p \int_{\Omega} \vec{u} \cdot \nabla T \varphi \mathrm{d}x + k \int_{\Omega} \nabla T \cdot \nabla \varphi \mathrm{d}x + \int_{\Gamma_{\mathrm{amb}}} \left(h_{\mathrm{amb}} T \mathrm{d}\sigma + \sigma \varepsilon T^4\right) \varphi \mathrm{d}\sigma + \int_{\Gamma_{\mathrm{body}}} h_{\mathrm{bl}} T \varphi \mathrm{d}\sigma$$

$$= \int_{\Gamma_{\mathrm{amb}}} \left(h_{\mathrm{amb}} T_{\mathrm{amb}} + \sigma \varepsilon T_{\mathrm{amb}}^4\right) \varphi \mathrm{d}\sigma + \int_{\Gamma_{\mathrm{body}}} h_{\mathrm{bl}} T_{\mathrm{bl}} \varphi \mathrm{d}\sigma.$$

Weak formulation

Find $(\vec{u}, p, T) \in \boldsymbol{V} \times Q \times W$ such that for all $(\vec{v}, q, \varphi) \in \boldsymbol{V} \times Q \times W$:

$$\begin{aligned} a_1(\vec{u}, \vec{u}, \vec{v}) + a_2(\vec{u}, \vec{v}) + b(p, \vec{v}) + d(T, \vec{v}) &= \ell_1(\vec{v}), \\ b(q, \vec{u}) \qquad\qquad\qquad\qquad\qquad &= 0, \\ e(\vec{u}, T, \varphi) \qquad\qquad\qquad + f(T, \varphi) &= \ell_2(\varphi). \end{aligned}$$

Weak formulation

Find $(\vec{u}, p, T) \in \boldsymbol{V} \times Q \times W$ such that for all $(\vec{v}, q, \varphi) \in \boldsymbol{V} \times Q \times W$:

$$\begin{aligned} a_1(\vec{u}, \vec{u}, \vec{v}) + a_2(\vec{u}, \vec{v}) + b(p, \vec{v}) + d(T, \vec{v}) &= \ell_1(\vec{v}), \\ b(q, \vec{u}) &= 0, \\ e(\vec{u}, T, \varphi) \qquad\qquad + f(T, \varphi) &= \ell_2(\varphi). \end{aligned}$$

- $\boldsymbol{V}_h := \left\{ \vec{v} \in [\mathbb{P}_2(\Omega_{\mathrm{AH}})]^3 \,\middle|\, \vec{v} = \vec{0} \text{ on } \partial\Omega \right\}$, the space of vector-valued piecewise quadratic polynomials that vanish on the boundary for velocity;
- $Q_h := \mathbb{P}_1(\Omega_{\mathrm{AH}})$, the space of piecewise linear polynomials for pressure; and
- $W_h := \mathbb{P}_1(\Omega)$, the space of piecewise linear polynomials for temperature.

Weak formulation

The algebraic problem to solve is then :

$$\begin{bmatrix} \underline{\underline{\boldsymbol{V}}}^k + \underline{\underline{\boldsymbol{W}}}^k + \underline{\underline{\boldsymbol{N}}} & \underline{\underline{\boldsymbol{B}}}^T & \underline{\underline{\boldsymbol{D}}} \\ \underline{\underline{\boldsymbol{B}}} & \underline{\underline{\boldsymbol{0}}} & \underline{\underline{\boldsymbol{0}}} \\ \underline{\underline{\boldsymbol{E}}}_1^k & \underline{\underline{\boldsymbol{0}}} & \underline{\underline{\boldsymbol{E}}}_2^k + \underline{\underline{\boldsymbol{F}}} \end{bmatrix} \begin{bmatrix} \underline{\boldsymbol{\Delta u}} \\ \underline{\boldsymbol{\Delta p}} \\ \underline{\boldsymbol{\Delta T}} \end{bmatrix} = \begin{bmatrix} \underline{\boldsymbol{r}}_{\vec{u}}^k \\ \underline{\boldsymbol{r}}_p^k \\ \underline{\boldsymbol{r}}_T^k \end{bmatrix},$$

Problem of dimension $\dim(\boldsymbol{V}_h) + \dim(Q_h) + \dim(W_h)$.
In this example :

- $\dim(\boldsymbol{V}_h) = 2.42 \cdot 10^5$
- $\dim(Q_h) = 7.08 \cdot 10^5$
- $\dim(W_h) = 34{,}304$

Pressure and velocity: impact of the posture

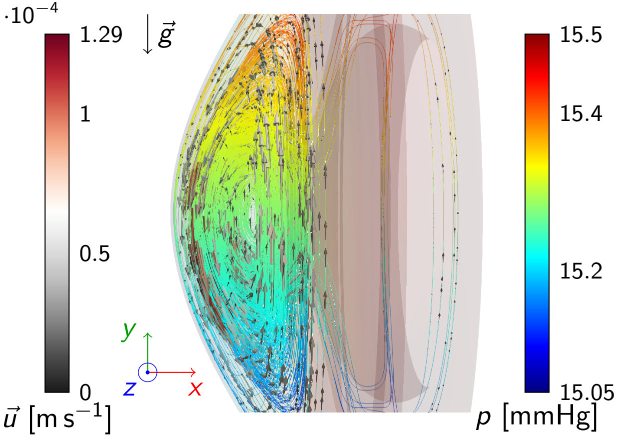


Figure: Standing position.

- **Recirculation** of the AH,

[a] Wang *et al.* BioMedical Engineering OnLine. (2016)

[b] Abdelhafid *et al.* *Recent Devel. in Mathematical, Statistical and Computational Sciences.* (2021)

[c] Murgoitio-Esandi *et al.* *Translational Vision Science & Technology.* (2023)

Pressure and velocity: impact of the posture

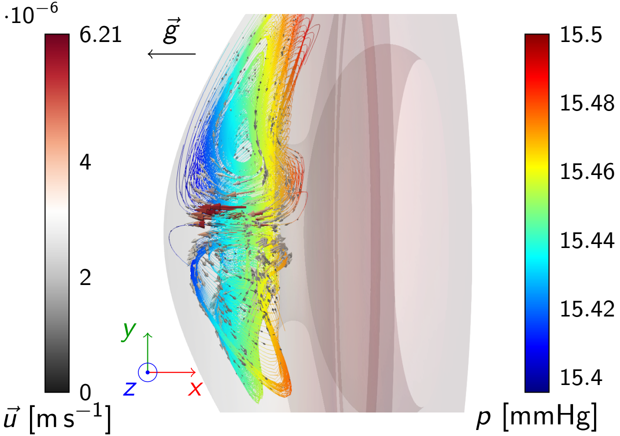


Figure: Prone position.

- **Recirculation** of the AH,
- Formation of a **Krukenberg's spindle**, in good agreement with clinical observations and previous studies[a,b,c]

[a] Wang *et al.* BioMedical Engineering OnLine. (2016)

[b] Abdelhafid *et al.* *Recent Devel. in Mathematical, Statistical and Computational Sciences.* (2021)

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Pressure and velocity: impact of the posture

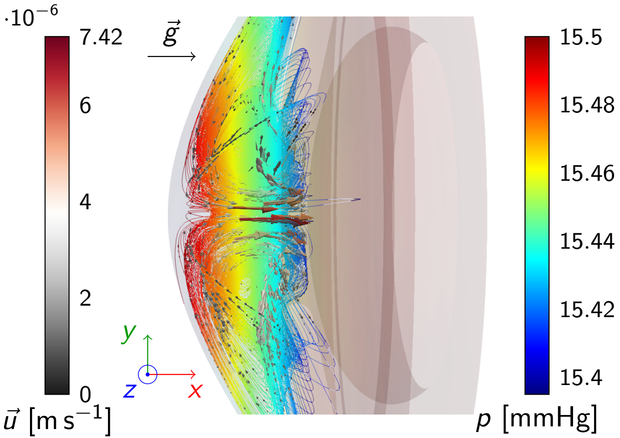

Figure: Supine position.

- **Recirculation** of the AH,
- Formation of a **Krukenberg's spindle**, in good agreement with clinical observations and previous studies[a][b][c]
- Fluid dynamics is **strongly influenced by the position of the patient**.

[a] Wang *et al.* BioMedical Engineering OnLine. (2016)
[b] Abdelhafid *et al.* *Recent Devel. in Mathematical, Statistical and Computational Sciences.* (2021)
[c] Murgoitio-Esandi *et al.* *Translational Vision Science & Technology.* (2023)

Thank you for your attention!

Any question ?